A General Semi-Lagrangian Advection Scheme Employing Forward Trajectories

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Abstract

We propose a semi-Lagrangian advection scheme by using forward trajectories and a general interpolation procedure from the irregularly distributed Lagrangian grid to the regularly distributed Eulerian grid. The general interpolation procedure has virtually no restrictions, and the advection scheme is absolutely stable due to the direct calculation of forward trajectories. They are, therefore, applicable to problems of any extent of deformation with arbitrary Courant numbers. The scheme is also quite efficient, because this interpolation procedure actually converts a two-dimensional problem into a sequence of one-dimensional problems by employing an intermediate grid. The slotted cylinder of Zalesak under the motion of solid-body rotation is tested for the accuracy of this 'internet interpolation method'. The idealized cyclogenesis of Doswell is simulated to demonstrate the ability of the advection scheme for accommodating strong deformation flows. Both tests indicate satisfactory accuracy.

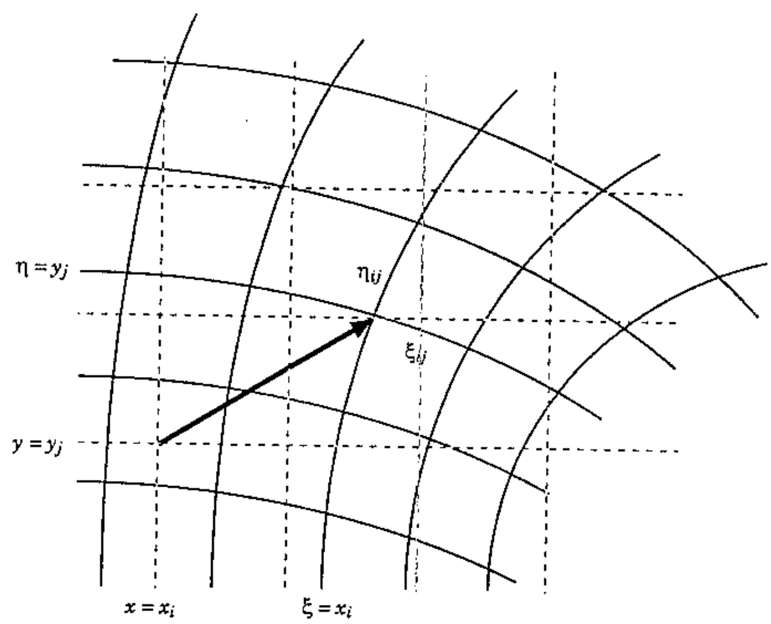

Figure 2. A particle on the Eulerian grid (dashed lines) at time t_n travels to a point on the Lagrangian grid (full lines) at time t_{n+1} as shown by the arrow, and the corresponding Eulerian coordinate lines $x = x_i$ and $y = y_j$ are transformed to the Lagrangian coordinate lines $x = x_i$ and $h = y_j$, respectively. We note that the values of f on the Eulerian grid can be easily interpolated with one-dimensional formulae from the intermediate grid consisting of the intersections of the Lagrangian curves and the Eulerian y-lines $x = x_i$.

1. INTRODUCTION

Backward trajectories are currently widely adopted in semi-Lagrangian schemes. Their main advantage is that the variables can be evaluated conveniently on a straight-line Eulerian grid; the disadvantage is that trajectories cannot be calculated directly. Instead, implicit equations need to be solved iteratively for the displacements, and this can be very costly. On the other hand, forward trajectories can be obtained easily with great accuracy (Purser and Leslie 1994), but the variables are difficult to evaluate on a curvilinear Lagrangian grid. One of the major difficulties in using forward trajectories for semi-Lagrangian advection is how to interpolate from the irregularly distributed Lagrangian grid to the regularly distributed Eulerian grid. Purser and Leslie (1991) found the 'cascade interpolation method', and Sun *et al.* (1996) proposed the 'split interpolation method', both methods being based on splitting the dimensions of the geometric space to simplify the interpolation formulas. These two methods are both very efficient with satisfactory accuracy, but the dimension split leads to a restriction which is slightly more stringent than the stability criterion of conventional semi-Lagrangian schemes (Sun *et al.* 1996). By using the intersections of the curvilinear Lagrangian network and the straightline Eulerian network, we have found an accurate and efficient interpolation method without splitting the dimensions, and therefore avoid the associated restriction. It is called the 'internet interpolation method', because an intermediate grid consisting of the intersections is employed in the procedure, as is presented in Section 2.

2. INTERNET INTERPOLATION METHOD

Consider the problem of two-dimensional passive advection which is governed by the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0, \tag{1}$$

Assume that we know $f(x, y, t)$ at all Eulerian grid points at time t_n; we wish to obtain values at all Eulerian grid points at time t_{n+1}. The semi-Lagrangian advection employing forward trajectories is:

$$\frac{f(x_i + \alpha_{ij}, y_j + \beta_{ij}, t_{n+1}) - f(x_i, y_j, t_n)}{\Delta t} = 0,$$

(2)

where a_{ij} and b_{ij} are the x- and y-components of the displacement of the (i, j)th particle within the time interval $[t_n, t_{n+1}]$, respectively. They can be calculated directly by the integrals

$$\alpha_{ij} = \int_{t_n}^{t_{n+1}} u_{ij}(x, y, t)dt,$$

$$\beta_{ij} = \int_{t_n}^{t_{n+1}} v_{ij}(x, y, t)dt,$$

Finally, we need to interpolate the values of f from the Lagrangian grid to the Eulerian grid. Given an Eulerian coordinate system, let (x_i, y_j) be the Eulerian coordinates of the (i, j)th Eulerian grid point. We notice that the lines defined by the equations $x = x_i$ and $y = y_j$ constitute an *Eulerian network.* On the other hand, a *Lagrangian network* is induced by the motion of the fluid from the Eulerian network, as illustrated in Fig. 1. We note that the particle that is located at the (i, j)th Eulerian grid point at time t_n travels to the (i, j)th Lagrangian grid point at time t_{n+1}, and a natural Lagrangian coordinate system can be defined such that the Lagrangian coordinates of the (i, j)th Lagrangian grid point are just (x_i, y_j). Furthermore, the Eulerian coordinates of the (i, j)th Lagrangian grid point are just $X_{ij} = x_i + a_{ij}$ and $Y_{ij} = y_j + b_{ij}$. Consequently, the Lagrangian network can be well constructed by fitting curves to the data $\{ X_{ij} \}$, $\{ Y_{ij} \}$, $\{ f_{ij} \}$ on the Lagrangian coordinate system. The Lagrangian network thus provides information about the Eulerian coordinates and the values of f for an arbitrary point with given Lagrangian coordinates.

The strategy of the internet interpolation method is to find an intermediate grid consisting of the intersections of the Lagrangian network and the Eulerian network, and then interpolate from the intermediate grid to the Eulerian grid. To find the intersections and, therefore, the intermediate grid in terms of Eulerian coordinates, let us consider the x-curve x_{ij} in the Lagrangian network connecting the (i, j)th and the $(i+1, j)$th Lagrangian grid points. Assume that the Lagrangian network is constructed with cubic polynomials, then the Eulerian coordinates and the values of f for the points on the Lagrangian x-curve x_{ij} are expressed as

$$X(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 , \tag{3}$$

$$Y(s) = b_3 s^3 + b_2 s^2 + b_1 s + b_0 , \tag{4}$$

$$F(s) = c_3 s^3 + c_2 s^2 + c_1 s + c_0 , \tag{5}$$

with the Lagrangian coordinates s Œ $[x_i , x_{i+1}]$. Moreover, the range of $X(s)$ can be easily determined by finding its critical points, i.e. where $dX/ds = 0$. Suppose that x_k is in the range of $X(s)$, then the Lagrangian x-curve x_{ij} intersects the Eulerian y-line l_k defined by the equation $x = x_k$ in terms of Eulerian coordinates. Solving the equation

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0 = x_k \tag{6}$$

for s Œ $[x_i , x_{i+1}]$ by formulas, we can obtain at most three distinct roots: s_1, s_2 and s_3. Substituting the roots into Eqs. (4) and (5) yields three intersection points $(x_k , Y(s_1))$, $(x_k , Y(s_2))$ and $(x_k , Y(s_3))$ in terms of Eulerian coordinates, with the f-values $F(s_1)$, $F(s_2)$ and $F(s_3)$, respectively. Similarly, we can find the intersections of the Eulerian y-line l_k and the Lagrangian y-curve h_{ij} that connects the (i, j)th and the $(i, j+1)$th Lagrangian grid points. We collect all the intersections of the Lagrangian curves $\{ x_{ij} , h_{ij} \}$ and the Eulerian y-lines $\{ l_k \}$, and sort the data points on Eulerian y-coordinates for each Eulerian y-line, then obtain the intermediate grid in terms of Eulerian coordinates. The values of f on the Eulerian grid can therefore be easily interpolated from the intermediate grid with one-dimensional formulas on the Eulerian y-lines, as illustrated in Fig. 2. We summarize the procedure of the internet interpolation method as:

(i) Construct the Lagrangian network induced by the motion of the fluid from the Eulerian network, by fitting the data $\{ X_{ij} \}$, $\{ Y_{ij} \}$, $\{ f_{ij} \}$ on the Lagrangian coordinate system with cubic or other polynomials.

(ii) Find the intermediate grid consisting of the intersections of the Lagrangian curves $\{ x_{ij} , h_{ij} \}$ and the Eulerian y-lines $\{ l_k \}$ in terms of Eulerian coordinates,using Eqs. (3), (4) and (5).

(iii) Interpolate the values of f from the intermediate grid onto the Eulerian grid with one-dimensional formulas on the Eulerian y-lines.

3. NUMERICAL SIMULATIONS

To emphasize the smoothness of the flows, cubic splines are employed for all interpolations in the tests. For properties of the Lagrange polynomials of degrees 3, 5 and 7 incorporated in the advection scheme employing forward trajectories, the readers are referred to Sun *et al.* (1996). The spatial filter of Sun *et al.* (1996) is adopted to eliminate the spurious short waves and to achieve the positive-definiteness. Exact trajectories are used in all tests, because the velocity fields are steady-state and the trajectories can be integrated exactly. Analytic boundary conditions are used in all tests. The maximum Courant number is defined as

CN = max { | u Dt /Dx |, | v Dt /Dy | }.

The accuracy is measured by the root-mean-square error definedas

$$\text{Error} = \left\{ \frac{1}{\text{NM}} \sum_{i=1}^{\text{N}} \sum_{j=1}^{\text{M}} (f_{ij} - g_{ij})^2 \right\}^{1/2},$$

where f_{ij} and g_{ij} are the numerical and analytic solutions, respectively, and N and M are the numbers of grid points in the x- and y-directions, respectively.

(a) Slotted cylinder under solid-body rotation

The accuracy of the internet interpolation method is tested using the slotted cylinder of Zalesak (1979) under the motion of solid-body rotation about the center of the domain. The initial condition is shown in Fig. 3(a), where the radius of the cylinder is 15 Δx, the width of the groove is 6 Δx, the domain is 100¥100 Δx^2, and the amplitude of the distribution is 1. The numerical solution after five revolutions (220 time steps, CN = 7.1, Error = 0.061), obtained using the internet interpolation method, is almost identical to that obtained by the split interpolation method (see Figure 8 of Sun *et al.* 1996). Another comparison is made to the excellent results of Bermejo and Staniforth (1992). Our numerical solution after six revolutions (576 time steps, CN = 3.3, Error = 0.068) is shown in Fig. 3(b). We see that the shape is well preserved--the amplitude is well maintained, the groove is well resolved and the dispersion is negligible. A profile across the groove is shown in Fig. 3(c).

(b) Doswell's idealized cyclogenesis

The idealized cyclogenesis of Doswell (1984) is simulated to demonstrate the ability of the advection scheme for strong deformation flows. The velocity field is a circular vortex with a tangential velocity

$V(r) = A\ \text{sech}^2 (r) \tanh (r)$,

where r is the radius of the vortex and A is chosen so that the maximum value of V equals 1. The scalar initial condition is

$f(x, y, 0) = -\tanh [(y - y_c)/\text{d}]$,

where d is the characteristic width of the front zone, as defined by Hólm (1995). The analytic solution is

$$f(x, y, t) = -\tanh\left[\frac{(y - y_c)}{\delta} \cos(\omega t) - \frac{(x - x_c)}{\delta} \sin(\omega t) \right],$$

where (x_c, y_c) is the center of rotation and w = V/r is the angular velocity.

The integration domain is 10 units long with a resolution of 65^2 and 129^2 grid points. The integration time is 5 units, and is chosen so that the analytic solution is still resolvable on the low resolution grid. Nearly perfect results are obtained for the smooth cases with d ª 1, and we are interested only in the non-smooth case with d = 0.05. The analytic solution on the high resolution grid is shown in Fig. 4(a). We note that the simulation (64 time steps, CN = 1, Error = 0.083) shown in Fig. 4(b) is comparable to the excellent results of Hólm (1995). The advantage of our scheme, however, is that there is no restriction on the maximum Courant number. A simulation of a greater maximum Courant number (16 time steps, CN = 4, Error = 0.068) is shown in Fig. 4(c). We also note that almost-identical results are obtained on the high resolution grid when the economic version of the intermediate grid is employed.

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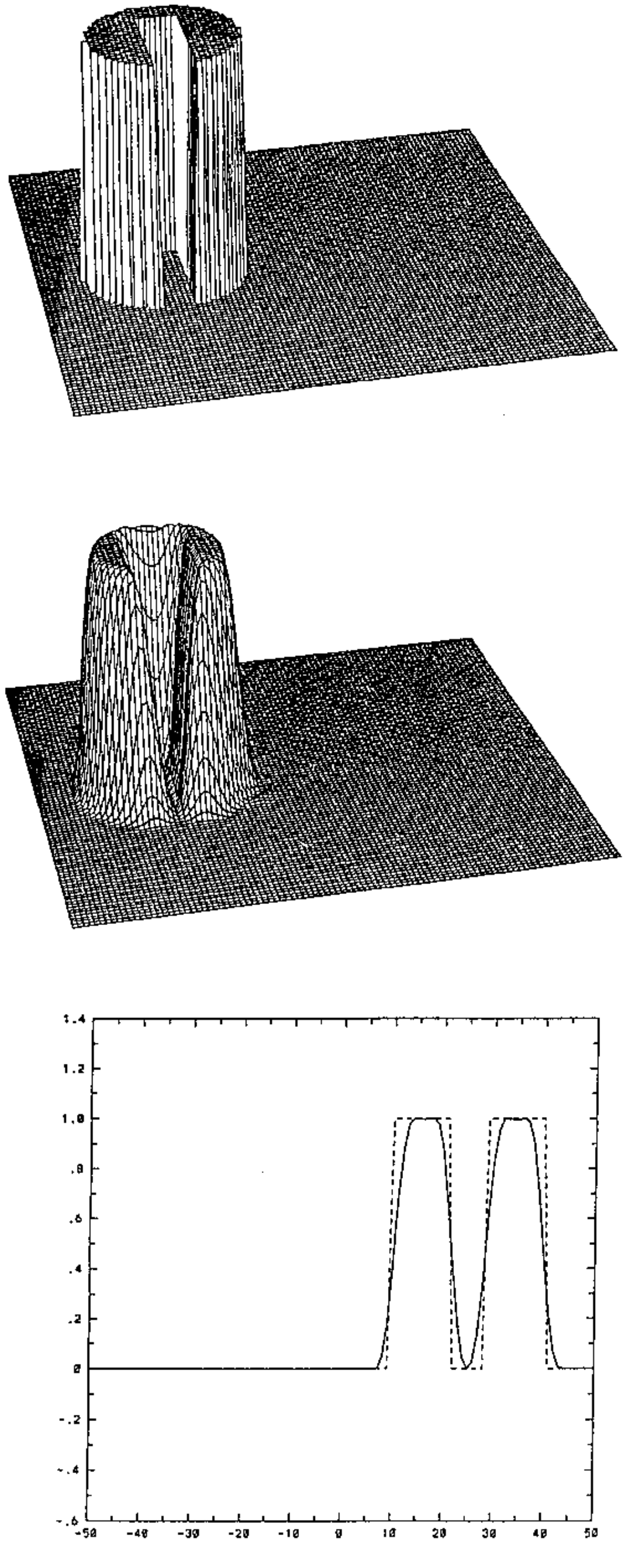

Figure 3. The slotted cylinder test. (a) The initial condition. (b) The numerical solution after six revolutions (576 time steps, CN = 3.3, Error = 0.068). (c) The profile of the numerical solution across the groove through the center of domain, where the analytic and numerical solutions are shown by dashed and solid lines, respectively.

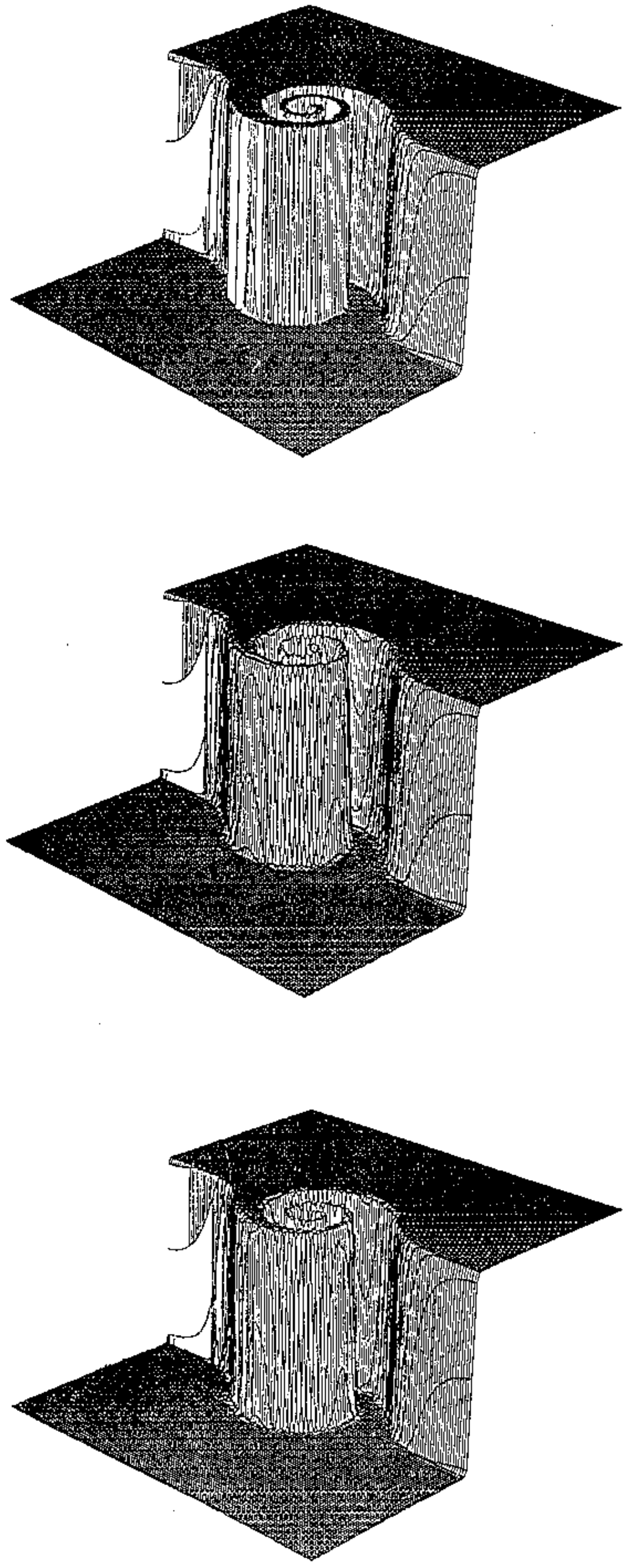

Figure 4. The idealized cyclogenesis test in the non-smooth case. (a) The analytic solution after 5 time units. (b) The numerical solution at the same time with 64 time steps (CN = 1, Error = 0.083). (c) The numerical solution at the same time with 16 time steps (CN = 4, Error = 0.068).